Journal of Sound and Vibration (2001) **248**(3), 413–426
doi:10.1006/jsvi.2000.3826, available online at http://www.idealibrary.com on IDEAL®

QUANTITATIVE MEASURES OF MODAL CONTROLLABILITY AND OBSERVABILITY IN VIBRATION CONTROL OF DEFECTIVE AND NEAR-DEFECTIVE SYSTEMS

Y. D. CHEN, S. H. CHEN AND Z. S. LIU

Department of Mechanics, Jilin University, Nanling Campus, Changchun 130025, *People's Republic of China. E-mail*: *chensh@jlu.eud.cn*

(*Received* 26 *January* 2000, *and in final form* 22 *December* 2000)

In this paper, we focus on the control problem of defective systems with repeated eigenvalues. Using the singular-value decomposition of input matrix **B** and output matrix **C** in the modal subspace, the quantitative measures of modal controllability and observability of defective systems are discussed. For a near-defective system with close eigenvalues, we first transform it into a defective one, and then apply the same method to deal with near-defective systems. Numerical examples for a general damping system and an airfoil are given to illustrate the applications of the present method.

1. INTRODUCTION

The modal controllability and observability for systems with distinct or repeated eigenvalues of non-defective systems have been well discussed. For example, reference [1] investigated the modal controllability and observability for distinct modes; references [2–4] discussed criteria for the modal controllability and observability of repeated modes. The singular-value decomposition (SVD) technique was widely used to deal with the control problems. Recent papers in this field include the measures of modal controllability and observability for non-defective systems with repeated eigenvalues [5, 6], indirect adaptive periodic control [7], controllability analysis of thermally coupled distillation systems [8], stability and modal control for multifingers robot band [9], the controllability and observability analysis for adaptive optical systems [10], and the index of controllability and observability of a given mode [11].

The above discussions mainly involve the control problems of the non-defective system, which has a set of complete eigenvectors to span the eigenspace. However, in actual engineering problems, such as general damping systems, flutter analysis of aeroelasticity, and so on, the system called "defective system" does not have a set of complete eigenvectors to span the eigenspace [12]. In this special case, the state matrix A cannot be diagonalized. It is well known that the defective system with repeated eigenvalues is ill-conditioned because the dynamic characteristic is very sensitive to the changes of parameters of the defective system with repeated eigenvalues, and it can be changed into a near-defective system with close eigenvalues [13]. Therefore, the difficulty arises in designing the modal control of the defective or near-defective system. The major difficulty is that the generalized

0022-460X/01/480413 + 14 $35.00/0

right and left modal matrices, **U** and **V**, cannot be obtained with the standard methods for extracting the modal matrix, and that from the view point of mathematics, the close eigenvalues of near-defective systems are distinct, but the dynamic characteristic is still defective. Therefore, it is necessary to discuss the modal control problems of the defective and near-defective systems.

This study focuses on the quantitative measures of the modal controllability and observability for a defective system with repeated eigenvalues based on the modal control equations, and also for a near-defective system with close eigenvalues. For a near defective system, we first transform it into a defective one, and then apply the same method to deal with the near-defective system. The theory is illustrated by numerical examples for a general damping system and an airfoil to prove the validity.

2. GENERALIZED MODAL THEORY OF DEFECTIVE SYSTEMS

The generalized modal theory can be found in reference [12]. Consider a linear vibration system described by

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{H})\mathbf{x} = 0, \tag{1}$$

where it is assumed that **M**, **D**, **G**, **K**, and **H** are real matrices, **M**, **D**, **K** are symmetric matrices, $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$, $\mathbf{D} = \mathbf{D}^{\mathrm{T}}$, $\mathbf{K} = \mathbf{K}^{\mathrm{T}}$, corresponding to the mass, damping, and stiffness respectively, **G** and **H** are skew-symmetric matrices, $\mathbf{G}^{\mathrm{T}} = -\mathbf{G}$, $\mathbf{H}^{\mathrm{T}} = -\mathbf{H}$, corresponding to gyroscopic and circulatory (or non-conservative positional) forces, respectively, and **M** is assumed to be positive definite. The eigenvalue problem is as follows:

$$(\mathbf{M}\lambda^2 + (\mathbf{D} + \mathbf{G})\lambda + (\mathbf{K} + \mathbf{H}))\mathbf{x} = 0. \tag{2}$$

Using the state vector

$$\mathbf{u} = \begin{bmatrix} \lambda\mathbf{x} \\ \mathbf{x} \end{bmatrix}, \tag{3}$$

one has

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}, \tag{4}$$

where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}(\mathbf{D} + \mathbf{G}) & -\mathbf{M}^{-1}(\mathbf{K} + \mathbf{H}) \\ \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{5}$$

In equation (5), **I** is the unit matrix and **0** is the zero matrix of the same order.

It is assumed that **AM** is used to denote the algebra multiplicity of the eigenvalue λ in equation (4), and **GM** is used to denote the number of the linear independent eigenvectors corresponding to λ. If **AM** = **GM** for the distinct and repeated eigenvalues, the system is non-defective; if AM > GM, the system with repeated eigenvalues is defective.

From the algebra theory for the defective matrix **A**, there exists non-singular matrix **U**, such that

$$\mathbf{AU} = \mathbf{UJ}, \tag{6}$$

where **U** is the generalized modal matrix of **A**, **J** is the Jordan block of **A** given by

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ & & & \mathbf{J}_r \end{bmatrix}, \tag{7}$$

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{m_i \times m_i} \qquad \sum_{i=1}^{r} m_i = n. \tag{8}$$

Equation (6) can be written in the following manner:

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{u}_1^{(i)} = 0, \qquad (\mathbf{A} - \lambda_i \mathbf{I})\mathbf{u}_j^{(i)} = \mathbf{u}_{j-1}^{(i)}, \quad j = 2, 3, \ldots, m_i, \quad i = 1, 2, \ldots, r, \tag{9}$$

where $\mathbf{u}_j^{(i)}$ denotes the jth generalized mode corresponding to the ith defective eigenvalue λ_i. The conjugate and transpose of **A** is called adjoint system, i.e., for $\mathbf{A}^{\mathrm{H}}$ the generalized modes satisfy the following equation:

$$\mathbf{A}^{\mathrm{H}}\mathbf{V} = \mathbf{V}\mathbf{J}^{\mathrm{H}}, \tag{10}$$

where $\mathbf{A}^{\mathrm{H}}$ and $\mathbf{J}^{\mathrm{H}}$ are the conjugate and transpose of **A** and **J**, respectively, **V** is the generalized modal matrix of $\mathbf{A}^{\mathrm{H}}$.

Equation (10) can also be written as follows:

$$(\mathbf{A}^{\mathrm{H}} - \tilde{\lambda}_i \mathbf{I})\mathbf{v}_j^{(i)} = \mathbf{v}_{j+1}^{(i)}, \quad j = 1, 2, 3, \ldots, m_{i-1},$$

$$(\mathbf{A}^{\mathrm{H}} - \tilde{\lambda}_i \mathbf{I})\mathbf{v}_{mi}^{(i)} = 0, \qquad i = 1, 2, \ldots, r, \tag{11}$$

where $\tilde{\lambda}_i$ is the conjugate of λ_i. In general, $\mathbf{u}_i (i = 1, 2, \ldots, r)$ are known as the right eigenvectors, $\mathbf{v}_i (i = 1, 2, \ldots, r)$ are known as the left eigenvectors, $\mathbf{u}_{i+1}, \ldots, \mathbf{u}_{i+m_i-1}$ and $\mathbf{v}_{i+1}, \ldots, \mathbf{v}_{i+m_i-1}$ are the right and left generalized modes corresponding to λ_i respectively.

The right generalized modal matrix **U** and the left generalized modal matrix **V** satisfy the following orthogonal condition:

$$\mathbf{V}^{\mathrm{H}}\mathbf{U} = \mathbf{I}. \tag{12}$$

3. QUANTITATIVE MEASURES FOR MODAL CONTROLLABILITY AND OBSERVABILITY OF DEFECTIVE SYSTEMS WITH REPEATED EIGENVALUES

We consider the control system indicated by the following state equation:

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{Z}(t), \qquad \mathbf{y}(t) = \mathbf{C}\mathbf{X}(t), \tag{13}$$

where the state matrix is given by equation (5), $\mathbf{X}(t) \in \mathbf{R}^{n \times 1}$ is the state vector, $\mathbf{Z}(t) \in \mathbf{R}^{r \times 1}$ is the input vector, $\mathbf{y}(t) \in \mathbf{R}^{q \times 1}$ is the output vector, matrices $\mathbf{B} \in \mathbf{R}^{n \times r}$ and $\mathbf{C} \in \mathbf{R}^{q \times n}$ are called the actuator distribution matrix and sensor distribution matrix, respectively, indicating the locations of control forces and sensors.

Transforming equation (13) into the modal co-ordinates through the modal transformation

$$\mathbf{X}(t) = \mathbf{U}\xi(t) \tag{14}$$

yields

$$\begin{bmatrix} \dot{\xi}_m \\ \cdots \\ \dot{\xi}_d \end{bmatrix} = \begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_d \end{bmatrix} \begin{bmatrix} \xi_m \\ \cdots \\ \xi_d \end{bmatrix} + \begin{bmatrix} \mathbf{V}_m^{\mathrm{H}} \\ \cdots \\ \mathbf{V}_d^{\mathrm{H}} \end{bmatrix} \mathbf{BZ}(t) \tag{15}$$

and

$$\mathbf{y}(t) = \mathbf{C}[U_m, U_{n-m}] \begin{bmatrix} \xi_m \\ \cdots \\ \xi_d \end{bmatrix} = \mathbf{CU}_m\xi_m(t) + \mathbf{CU}_{n-m}\xi_d(t), \tag{16}$$

where

$$\mathbf{J} = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{m \times m}, \tag{17}$$

$$\mathbf{\Lambda}_d = \begin{bmatrix} \lambda_{m+1} & & & 0 \\ & \lambda_{m+2} & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}. \tag{18}$$

In equation (15), we assume that $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \lambda$ are defective repeated eigenvalues with m multiplicity, and the rest of the eigenvalues, $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n$, are distinct, the right and the left modal matrices are expressed as the partitional form, $\mathbf{U} = [\mathbf{U}_m, \mathbf{U}_{n-m}]$, $\mathbf{V} = [\mathbf{V}_m, \mathbf{V}_{n-m}]$, ξ_m and ξ_d are the modal co-ordinates corresponding to the repeated and distinct eigenvalues.

From equations (15) and (16), we obtain the control governing equations corresponding to the defective repeated eigenvalues and distinct eigenvalues in terms of the generalized modal co-ordinates

$$\dot{\xi}_m = \mathbf{J}\xi_m + \mathbf{V}_m^{\mathrm{H}}\mathbf{BZ}(t), \qquad \dot{\xi}_d = \mathbf{\Lambda}_d\xi_d + \mathbf{V}_{n-m}^{\mathrm{H}}\mathbf{BZ}(t), \tag{19, 20}$$

$$\mathbf{y}_m = \mathbf{CU}_m\xi_m, \qquad \mathbf{y}_d = \mathbf{CU}_{n-m}\xi_d. \tag{21, 22}$$

Equations (19) and (21) can be written as

$$\dot{\xi}_m = \mathbf{J}\xi_m + \mathbf{P}_m\mathbf{Z}(t), \qquad \mathbf{y}_m = \mathbf{C}_m\xi_m, \tag{23, 24}$$

where

$$\mathbf{P}_m = \mathbf{V}_m^{\mathrm{H}}\mathbf{B}, \qquad \mathbf{C}_m = \mathbf{C}\mathbf{U}_m. \tag{25, 26}$$

In equations (25) and (26), $\mathbf{P}_m$ and $\mathbf{C}_m$ are called the modal controllable and observable matrices, respectively, which can be used to investigate the controllability and observability.

For the defective system with repeated eigenvalues expressed by equations (23) and (24), it is controllable if and only if [2]

$$rank(\mathbf{P}_m) = m \tag{27}$$

and it is observable if and only if

$$rank(\mathbf{C}_m) = m. \tag{28}$$

Reference [5] presented a criterion for the quantitative measures for modal controllability and observability of non-defective systems with repeated eigenvalues. In the following, we try to extend the idea presented in reference [5] to the case of the defective or near-defective systems with repeated or close eigenvalues.

Taking the SVD of the modal controllable matrix, $\mathbf{P}_m$ in equation (25), yields

$$\mathbf{P}_m = \Phi\mathbf{\Sigma}\Psi^{\mathrm{H}}, \tag{29}$$

where Φ is the right singular vectors of $\mathbf{P}_m$, $\Phi \in \mathbf{R}^{m\times m}$, Ψ is the left singular vectors of $\mathbf{P}_m$, $\Psi \in \mathbf{R}^{r\times r}$, $\Phi^{\mathrm{H}}\Phi = \mathbf{I}_m$, $\Psi^{\mathrm{H}}\Psi = \mathbf{I}_r$, and

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_p & \\ & 0 \end{bmatrix}_{m\times m}, \tag{30}$$

where $\mathbf{\Sigma}_p = \mathrm{diag}[\sigma_1, \sigma_2, \ldots, \sigma_p]$, and the non-zero positive numbers σ_i are in descending order, $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_p > 0$.

Similar to deductions presented in reference [5], the following equations can be obtained:

$$\dot{\eta} = \Phi^{\mathrm{H}}\mathbf{J}\Phi\eta + \mathbf{\Sigma}\mathbf{Z}'(t), \qquad \mathbf{Z}'(t) = \Psi^{\mathrm{H}}\mathbf{Z}(t), \tag{31, 32}$$

where η is the new generalized modal co-ordinates. From equation (31), it can be seen that the quantitative measure for controllability of each defective repeated mode can be shown clearly:

(1) Since the modal control vector $\mathbf{\Sigma}\mathbf{Z}_i'(t)$ is proportional to the singular-value σ_i, the controllability of η_i can be measured using σ_i. The greater σ_i is, the less the modal control force is needed to obtain the same control effect on η_i.
(2) The necessary and sufficient condition of the controllability of all the repeated eigenvalues of the defective system is $rank(\mathbf{\Sigma}) = m$, or $r \geqslant m$ and $p = m$.
(3) When $0 < p < m$, only a part of the repeated modes of the defective system is controllable, i.e., $\eta_i (i = 1, 2, \ldots, p)$ are controllable, leaving the rest, $\eta_i (p + 1 \leqslant i \leqslant m)$, uncontrollable.

On the basis of duality, we can also obtain the measures of modal observability of defective repeated eigenvalues.

4. QUANTITATIVE MEASURES FOR MODAL CONTROLLABILITY AND OBSERVABILITY OF NEAR-DEFECTIVE SYSTEMS WITH CLOSE EIGENVALUES

It should be pointed out that if some small changes of parameters of defective systems are introduced, the system with defective repeated eigenvalues can be perturbed into one with close eigenvalues, which is known as a near-defective system. For such a case, although the close eigenvalues are distinct, the dynamic characteristic of the system is still defective. Thus, the methods for dealing with controllability and observability of repeated eigenvalues of defective system as discussed in the above cannot be directly used to deal with a near-defective system with close eigenvalues. Therefore, it is necessary to discuss the quantitative measures of controllability and observability for a near-defective system with close eigenvalues.

Assume that n eigenvalues of $\mathbf{A}$ are close. The right modal matrix $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$, and the left modal matrix $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$, satisfy the following equations:

$$\mathbf{AU} = \mathbf{UJ}, \qquad \mathbf{A}^{\mathrm{H}}\mathbf{V} = \mathbf{VJ}^{\mathrm{H}} \tag{33, 34}$$

and the orthogonal condition

$$\mathbf{U}^{\mathrm{H}}\mathbf{V} = \mathbf{VU}^{\mathrm{H}} = \mathbf{I}, \tag{35}$$

where $\mathbf{J} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Taking the algebra average of $\lambda_1, \lambda_2, \ldots, \lambda_n$

$$\lambda_0 = \frac{1}{n}\sum_{i=1}^{n}\lambda_i \tag{36}$$

and letting

$$\mathbf{J}_0 = \begin{bmatrix} \lambda_0 & 1 & & \\ & \lambda_0 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{bmatrix} \tag{37}$$

and

$$\delta\mathbf{J}_0 = \begin{bmatrix} \lambda_1 - \lambda_0 & -1 & & \\ & \lambda_2 - \lambda_0 & \ddots & \\ & & \ddots & -1 \\ & & & \lambda_n - \lambda_0 \end{bmatrix}, \tag{38}$$

the matrix $\mathbf{J}$ in equation (33) can be written in the following form:

$$\mathbf{J} = \mathbf{J}_0 + \delta\mathbf{J}_0. \tag{39}$$

Considering the orthogonal condition (35) and substituting equation (39) into equation (33) yields

$$\begin{aligned} \mathbf{A} &= \mathbf{UJ}_0\mathbf{V}^{\mathrm{H}} + \mathbf{U}\delta\mathbf{J}_0\mathbf{V}^{\mathrm{H}} \\ &= \mathbf{A}_r + \delta\mathbf{A}, \end{aligned} \tag{40}$$

where

$$\mathbf{A}_r = \mathbf{U}\mathbf{J}_0\mathbf{V}^{\mathrm{H}}, \quad \delta\mathbf{A} = \mathbf{U}\delta\mathbf{J}_0\mathbf{V}^{\mathrm{H}}. \tag{41}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the close eigenvalues and $\delta = \max|\lambda_i - \lambda_0|$, it can be shown that the error matrix, $\delta\mathbf{A} = \mathbf{U}\delta\mathbf{J}_0\mathbf{V}^{\mathrm{H}}$, is a small perturbational one and its norm satisfies

$$\begin{aligned} \|\delta\mathbf{A}\|_2 &\leqslant \|\mathbf{U}\|_2\|\delta\mathbf{J}_0\|_2\|\mathbf{V}^{\mathrm{H}}\|_2 \\ &\leqslant \|\delta\mathbf{J}_0\|_2 \leqslant \delta^n. \end{aligned} \tag{42}$$

Since the eigenvalues of $\mathbf{J}_0$ cannot be changed by the orthogonal transformation, the eigenvalues of $\mathbf{A}_r$ are identical with those of $\mathbf{J}_0$. Equation (40) indicates that the matrix $\mathbf{A}$ is equal to the sum of the defective matrix $\mathbf{A}_r$ with n repeated eigenvalues and the perturbed matrix $\delta\mathbf{A}$ and the right and the left modal matrices $\mathbf{U}$ and $\mathbf{V}$ of $\mathbf{A}_r$ are the same as those of $\mathbf{A}$. Therefore, Equations (33) and (34) can be written as follows:

$$\mathbf{A}\mathbf{U} = \mathbf{U}(\mathbf{J}_0 + \delta\mathbf{J}_0), \qquad \mathbf{A}^{\mathrm{H}}\mathbf{V} = \mathbf{V}(\mathbf{J}_0 + \delta\mathbf{J}_0)^{\mathrm{H}}. \tag{43, 44}$$

Using the modal transformation

$$\mathbf{X}(t) = \mathbf{U}\xi(t), \tag{45}$$

the control equation (13) can be approximated by

$$\dot{\xi}(t) = \mathbf{J}_0\xi(t) + \mathbf{V}^{\mathrm{H}}\mathbf{B}\mathbf{Z}(t) \tag{46}$$

and

$$\mathbf{y}(t) = \mathbf{C}\mathbf{U}\xi(t). \tag{47}$$

Equations (46) and (47) show that the analysis for controllability and observability of the near-defective system with close eigenvalues can be transformed into one of the defective system with repeated eigenvalues which are equal to the average value of the close eigenvalues.

If the control equation of a near-defective system with close eigenvalues is expressed as one of the non-defective system with distinct eigenvalues

$$\dot{\xi}(t) = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\xi(t) + \mathbf{V}^{\mathrm{H}}\mathbf{B}\mathbf{Z}(t), \tag{48}$$

we have that since equation (48) represents a set of independent equations, the analysis results based on these equations will be misleading.

In addition, because the system is near defective, the right and left modal matrices $\mathbf{U}$ and $\mathbf{V}$ in equations (46) and (47) cannot be obtained using the following equations:

$$\mathbf{A}\mathbf{U} = \mathbf{U}\Lambda, \qquad \mathbf{A}^{\mathrm{H}}\mathbf{V} = \mathbf{V}\bar{\Lambda}, \tag{49, 50}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\bar{\Lambda}$ is the conjugate of Λ.

From the above discussions, it is known that for a near-defective system with close eigenvalues, we should use a suitable method with numerical stability such as the invariant subspace recursive method presented in reference [12] for computing the generalized modal matrix $\mathbf{U}$ and $\mathbf{V}$ based on equations (9) and (11), and the present method to obtain the controllability and observability of the defective system with repeated eigenvalues can be used to deal with the problem of near defective with close eigenvalues.

The invariant subspace recursive procedure for computing the generalized modes is given as follows:

(1) form **A** and compute $\lambda_i (i = 1, 2, \ldots, n)$ which are close eigenvalues;
(2) compute

$$\lambda_0 = \frac{1}{n} \sum_{i=1}^{n} \lambda_i;$$

(3) let $\mathbf{M} = \mathbf{A} - \lambda_0 \mathbf{I}$, $\Psi = \mathbf{I}$, $t = 0$;
(4) identify if $n = t$, turn to 9, otherwise, turn to 5;
(5) compute

$$\mathbf{M} = \Phi^{\mathrm{H}} \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_{t1} & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \Psi;$$

(6) compute

$$\mathbf{M} = \tilde{\Psi}^{\mathrm{H}} \mathbf{M} \tilde{\Psi}, \qquad \tilde{\Psi} = \begin{bmatrix} \Psi & \\ & I \end{bmatrix};$$

(7) record $\Psi = \Psi \tilde{\Psi}$;
(8) $t = t + t1$, turn to 4;
(9) Output Ψ, stop.

Here, Ψ is the right generalized modes **U** that we need. If $\mathbf{M} = \mathbf{A}^{\mathrm{H}} - \tilde{\lambda}_0 \mathbf{I}$, the left generalized modes **V** can be obtained. The details of invariant subspace recursive procedure can be found in reference [12].

5. NUMERICAL EXAMPLES

In order to illustrate the applications of the procedure presented in this paper, three numerical examples are given as follows.

Example 1. Consider a general damping system shown in Figure 1 with the parameter matrices given by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 4 & -2{\cdot}8284 \\ -2{\cdot}8284 & 6 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 36 & 0 \\ 0 & 81 \end{bmatrix}.$$

The state matrix **A** is

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{D} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & 0 \end{bmatrix} = \begin{bmatrix} -4 & 2{\cdot}8284 & -36 & 0 \\ 2{\cdot}8284 & -6 & 0 & -81 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The state vector is

$$\mathbf{x}(t) = [\dot{x}_1, \dot{x}_2, x_1, x_2]^{\mathrm{T}}$$

and the control matrix $\mathbf{B}$ in equation (13) for the single-input control force, $\mathbf{Z}(t)$, applied to mass 2 is

$$\mathbf{B} = \begin{bmatrix} 0 \\ \dots \\ \mathbf{M}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}$ are two pairs of 2-multiple conjugate roots

$$\lambda_1 = -2{\cdot}5 + 6{\cdot}910141\mathrm{i}, \qquad \lambda_2 = -2{\cdot}5 - 6{\cdot}910141\mathrm{i},$$

where $\mathrm{i} = \sqrt{-1}$

The Jordan form matrix is

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & 0 \\ 0 & \mathbf{J}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}.$$

This system is defective. By using the invariant subspace recursive procedure presented in the above section, the right and left modal matrices $\mathbf{U}$ and $\mathbf{V}$ can be obtained as follows:

$$\mathbf{U} = \begin{bmatrix} -0{\cdot}767520 + 0{\cdot}000000\mathrm{i} & 0{\cdot}565130 - 0{\cdot}255161\mathrm{i} & -0{\cdot}767520 + 0{\cdot}000000\mathrm{i} & 0{\cdot}565130 + 0{\cdot}255161\mathrm{i} \\ 0{\cdot}045226 - 0{\cdot}625048\mathrm{i} & -0{\cdot}265742 - 0{\cdot}723783\mathrm{i} & 0{\cdot}045226 + 0{\cdot}625048\mathrm{i} & -0{\cdot}265742 + 0{\cdot}723783\mathrm{i} \\ 0{\cdot}035533 + 0{\cdot}098216\mathrm{i} & -0{\cdot}088492 - 0{\cdot}091033\mathrm{i} & 0{\cdot}035533 - 0{\cdot}098216\mathrm{i} & -0{\cdot}088492 + 0{\cdot}091033\mathrm{i} \\ -0{\cdot}082078 + 0{\cdot}023150\mathrm{i} & -0{\cdot}053706 + 0{\cdot}045145\mathrm{i} & -0{\cdot}082078 - 0{\cdot}023150\mathrm{i} & -0{\cdot}053706 - 0{\cdot}045145\mathrm{i} \end{bmatrix}$$

and

$$\mathbf{V} = \begin{bmatrix} -0{\cdot}096825 + 0{\cdot}000000\mathrm{i} & -0{\cdot}010030 + 0{\cdot}130224\mathrm{i} & -0{\cdot}096825 + 0{\cdot}000000\mathrm{i} & -0{\cdot}010030 - 0{\cdot}130224\mathrm{i} \\ 0{\cdot}005705 + 0{\cdot}078851\mathrm{i} & -0{\cdot}080859 - 0{\cdot}000375\mathrm{i} & 0{\cdot}005705 - 0{\cdot}078851\mathrm{i} & -0{\cdot}080859 + 0{\cdot}000375\mathrm{i} \\ -0{\cdot}161374 + 0{\cdot}446047\mathrm{i} & 0{\cdot}788270 + 0{\cdot}355813\mathrm{i} & -0{\cdot}161374 - 0{\cdot}446047\mathrm{i} & 0{\cdot}788270 - 0{\cdot}355813\mathrm{i} \\ 0{\cdot}838700 + 0{\cdot}236553\mathrm{i} & -0{\cdot}164682 + 0{\cdot}448675\mathrm{i} & 0{\cdot}838700 - 0{\cdot}236553\mathrm{i} & -0{\cdot}164682 - 0{\cdot}448675\mathrm{i} \end{bmatrix}.$$

The modal control equations of this system are

$$\dot{\xi}_{m1} = \mathbf{J}_1\xi_{m1} + \mathbf{P}_{m1}z(t), \qquad \dot{\xi}_{m2} = \mathbf{J}_2\xi_{m2} + \mathbf{P}_{m2}z(t),$$

where

$$\mathbf{J}_1 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}, \qquad \mathbf{J}_2 = \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix}.$$

The control matrices in equation (25) corresponding to the two repeated eigenvalues subspaces are

$$\mathbf{P}_{m1} = \mathbf{V}_{m1}^{\mathrm{H}}\mathbf{B} = \begin{bmatrix} -0{\cdot}096825 + 0{\cdot}000000\mathrm{i} & -0{\cdot}010030 + 0{\cdot}130224\mathrm{i} \\ 0{\cdot}005705 + 0{\cdot}078851\mathrm{i} & -0{\cdot}080859 - 0{\cdot}000375\mathrm{i} \\ -0{\cdot}161374 + 0{\cdot}446047\mathrm{i} & 0{\cdot}788270 + 0{\cdot}355813\mathrm{i} \\ 0{\cdot}838700 + 0{\cdot}236553\mathrm{i} & -0{\cdot}164682 + 0{\cdot}448675\mathrm{i} \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0{\cdot}838700 & -0{\cdot}236553\mathrm{i} \\ -0{\cdot}164682 & -0{\cdot}448675\mathrm{i} \end{bmatrix},$$

$$\mathbf{P}_{m2} = \mathbf{V}_{m2}^{\mathrm{H}}\mathbf{B} = \begin{bmatrix} -0{\cdot}096825 + 0{\cdot}000000\mathrm{i} & -0{\cdot}010030 - 0{\cdot}130224\mathrm{i} \\ 0{\cdot}005705 - 0{\cdot}078851\mathrm{i} & -0{\cdot}080859 + 0{\cdot}000375\mathrm{i} \\ -0{\cdot}161374 - 0{\cdot}446047\mathrm{i} & 0{\cdot}788270 - 0{\cdot}355813\mathrm{i} \\ 0{\cdot}838700 - 0{\cdot}236553\mathrm{i} & -0{\cdot}164682 - 0{\cdot}448675\mathrm{i} \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0{\cdot}838700 + 0{\cdot}236553\mathrm{i} \\ -0{\cdot}164682 + 0{\cdot}448675\mathrm{i} \end{bmatrix}.$$

Taking the SVD of $\mathbf{P}_{m1}$ and $\mathbf{P}_{m2}$ in equation (29) yields

$$\mathbf{P}_{m1} = \Phi_1 \mathbf{\Sigma}_1 \Psi_1^{\mathrm{H}}, \quad \mathbf{P}_{m2} = \Phi_2 \mathbf{\Sigma}_2 \Psi_2^{\mathrm{H}}$$

$$\mathbf{\Sigma}_1 = \mathrm{diag}(\sigma_1^1 = 0{\cdot}993884, \sigma_2^1 = 0), \qquad \mathbf{\Sigma}_2 = \mathrm{diag}(\sigma_1^2 = 0{\cdot}993884, \sigma_2^2 = 0).$$

Since $\sigma_1^1 > 0$, $\sigma_2^1 = 0$, the first mode corresponding to the defective repeated eigenvalues λ_1 is controllable, and the second mode corresponding to λ_1 is uncontrollable. Similarly, the third mode corresponding to the defective repeated eigenvalue λ_2 is controllable, and the fourth mode corresponding to λ_2 is uncontrollable.

Example 2. Assume that the state matrix presented in Example 1 is perturbed into

$$\mathbf{A} = \begin{bmatrix} -4 & 2{\cdot}8284(1+\varepsilon) & -36 & 0 \\ 2{\cdot}8284 & -6 & 0 & -81 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

If the small parameter $\varepsilon = 0{\cdot}1 \times 10^{-3}$, the system has two sets of close eigenvalues, i.e., $\mathbf{J} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

$$\lambda_1 = -2{\cdot}512714 + 6{\cdot}905525\mathrm{i}, \qquad \lambda_2 = -2{\cdot}487286 + 6{\cdot}914724\mathrm{i},$$

$$\lambda_3 = -2{\cdot}512714 - 6{\cdot}905525\mathrm{i}, \qquad \lambda_4 = -2{\cdot}487286 - 6{\cdot}914724\mathrm{i}.$$

The right modes are

$$\mathbf{U}=\begin{bmatrix} -0{\cdot}767547+0{\cdot}000000i & 0{\cdot}565098-0{\cdot}255152i & -0{\cdot}767547+0{\cdot}000000i & 0{\cdot}565098+0{\cdot}255152i \\ 0{\cdot}045226-0{\cdot}625015i & -0{\cdot}265754-0{\cdot}723807i & 0{\cdot}045226+0{\cdot}625015i & -0{\cdot}265754+0{\cdot}723807i \\ 0{\cdot}035535+0{\cdot}098219i & -0{\cdot}088492-0{\cdot}091031i & 0{\cdot}035535-0{\cdot}098219i & -0{\cdot}088492+0{\cdot}091031i \\ -0{\cdot}082076+0{\cdot}023149i & -0{\cdot}053708+0{\cdot}045147i & -0{\cdot}082076-0{\cdot}023149i & -0{\cdot}053708-0{\cdot}045147i \end{bmatrix}$$

and the left modes are

$$\mathbf{V}=\begin{bmatrix} -0{\cdot}096821+0{\cdot}000000i & -0{\cdot}010048+0{\cdot}130223i & -0{\cdot}096821+0{\cdot}000000i & -0{\cdot}010048-0{\cdot}130223i \\ 0{\cdot}005706+0{\cdot}078852i & -0{\cdot}080860-0{\cdot}000386i & 0{\cdot}005706-0{\cdot}078852i & -0{\cdot}080860+0{\cdot}000386i \\ -0{\cdot}161368+0{\cdot}446030i & 0{\cdot}788229+0{\cdot}355930i & -0{\cdot}161368-0{\cdot}446030i & 0{\cdot}788229-0{\cdot}355930i \\ 0{\cdot}838711+0{\cdot}236553i & -0{\cdot}164738+0{\cdot}448635i & 0{\cdot}838711-0{\cdot}236553i & -0{\cdot}164738-0{\cdot}448635i \end{bmatrix}.$$

The above results show that the system is near defective.

The algebra average of λ_1, λ_2, is

$$\lambda_{01}=\frac{1}{2}\sum_{i=1}^{2}\lambda_i=-2{\cdot}5+6{\cdot}910124i.$$

and the algebra average of λ_3, λ_4, is

$$\lambda_{02}=\frac{1}{2}\sum_{i=3}^{4}\lambda_i=-2{\cdot}5-6{\cdot}910124i.$$

The Jordan form matrix of **A** is

$$\mathbf{J}=\mathbf{J}_0+\delta\mathbf{J}_0=\begin{bmatrix}\lambda_{01} & 1 & & \\ & \lambda_{01} & & \\ & & \lambda_{02} & 1 \\ & & & \lambda_{02}\end{bmatrix}+\begin{bmatrix}\lambda_1-\lambda_{01} & -1 & & \\ & \lambda_2-\lambda_{01} & -1 & \\ & & \lambda_3-\lambda_{02} & -1 \\ & & & \lambda_4-\lambda_{02}\end{bmatrix}.$$

Therefore, the near-defective system with close eigenvalues can be transformed into one of the defective system. With recursive procedure, the right and left generalized modes $\mathbf{U}=[\mathbf{U}_1, \mathbf{U}_2]$ and $\mathbf{V}=[\mathbf{V}_1, \mathbf{V}_2]$, corresponding to λ_{01} and λ_{02} can be obtained as follows:

$$\mathbf{U}_1=\begin{bmatrix} -0{\cdot}767547+0{\cdot}000000i & 0{\cdot}565098-0{\cdot}255152i \\ 0{\cdot}045226-0{\cdot}625015i & -0{\cdot}265754-0{\cdot}723807i \\ 0{\cdot}035535+0{\cdot}098219i & -0{\cdot}088492-0{\cdot}091031i \\ -0{\cdot}082076+0{\cdot}023149i & -0{\cdot}053708+0{\cdot}045147i \end{bmatrix},$$

$$\mathbf{U}_2=\begin{bmatrix} -0{\cdot}767547+0{\cdot}000000i & 0{\cdot}565098+0{\cdot}255152i \\ 0{\cdot}045226+0{\cdot}625015i & -0{\cdot}265754+0{\cdot}723807i \\ 0{\cdot}035535-0{\cdot}098219i & -0{\cdot}088492+0{\cdot}091031i \\ -0{\cdot}082076-0{\cdot}023149i & -0{\cdot}053708-0{\cdot}045147i \end{bmatrix},$$

$$\mathbf{V}_1 = \begin{bmatrix} -0{\cdot}096821 + 0{\cdot}000000\mathrm{i} & -0{\cdot}010048 + 0{\cdot}130223\mathrm{i} \\ 0{\cdot}005706 + 0{\cdot}078852\mathrm{i} & -0{\cdot}080860 - 0{\cdot}000386\mathrm{i} \\ -0{\cdot}161368 + 0{\cdot}446030\mathrm{i} & 0{\cdot}788229 + 0{\cdot}355930\mathrm{i} \\ 0{\cdot}838711 + 0{\cdot}236553\mathrm{i} & -0{\cdot}164738 + 0{\cdot}448635\mathrm{i} \end{bmatrix},$$

$$\mathbf{V}_2 = \begin{bmatrix} -0{\cdot}096821 + 0{\cdot}000000\mathrm{i} & -0{\cdot}010048 - 0{\cdot}130223\mathrm{i} \\ 0{\cdot}005706 - 0{\cdot}078852\mathrm{i} & -0{\cdot}080860 + 0{\cdot}000386\mathrm{i} \\ -0{\cdot}161368 - 0{\cdot}446030\mathrm{i} & 0{\cdot}788229 - 0{\cdot}355930\mathrm{i} \\ 0{\cdot}838711 - 0{\cdot}236553\mathrm{i} & -0{\cdot}164738 - 0{\cdot}448635\mathrm{i} \end{bmatrix}.$$

Taking the control matrices $\mathbf{P}_{m1}$ and $\mathbf{P}_{m2}$ in equation (46) corresponding to the two modal subspaces yields

$$\mathbf{P}_{m1} = \mathbf{V}_1^{\mathrm{H}}\mathbf{B} = \Phi_1\mathbf{\Sigma}_1\Psi_1^{\mathrm{H}}, \qquad \mathbf{P}_{m2} = \mathbf{V}_2^{\mathrm{H}}\mathbf{B} = \Phi_2\mathbf{\Sigma}_2\Psi_2^{\mathrm{H}},$$

where $\mathbf{\Sigma}_1 = \mathrm{diag}(\sigma_1^1 = 0{\cdot}993884, \sigma_2^1 = 0{\cdot}0)$, $\mathbf{\Sigma}_2 = \mathrm{diag}(\sigma_1^2 = 0{\cdot}993884, \sigma_2^2 = 0{\cdot}0)$

Since $\sigma_1^1 > 0$, $\sigma_2^1 = 0$, the first mode corresponding to λ_1 is controllable, the second mode is uncontrollable. Similarly, since $\sigma_1^2 > 0$, $\sigma_2^2 = 0$, the third mode is controllable, and the fourth mode is uncontrollable.

Example 3. We consider flutter problem of an airfoil in simplified formulation. The airfoil is replaced by a rigid rectangular panel with two degrees of freedom, a vertical displacement h and a rotation α. It is assumed that aerodynamic lift force is proportional to the angle of attack α and to the square of the velocity v of flight. The differential equations of motion are [14]

$$m\ddot{h} + s\ddot{\alpha} + K_h h = -\rho v^2 ab\alpha, \qquad s\ddot{h} + J_\alpha\ddot{\alpha} + K_\alpha\alpha = \rho v^2 abe\alpha.$$

If the parameters are given as follows: $m/(\rho ab^2) - 5$, $s/(mb) = 0{\cdot}25$, $J_\alpha/(mb^2) = 0{\cdot}5$, $e/b = 0{\cdot}4$, $K_h/m = 0{\cdot}25$, $K_\alpha/J_\alpha = 1$, and $u = v(J_\alpha/K_\alpha)^{1/2}/b$, then the above differential equations become

$$\mathbf{M}\ddot{x} + Kx = 0,$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & 0{\cdot}25 \\ 0{\cdot}25 & 0{\cdot}5 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 0{\cdot}25 & 0{\cdot}2u^2 \\ 0 & 0{\cdot}5 - 0{\cdot}08u^2 \end{bmatrix}.$$

If the parameter $u = 1{\cdot}32567735$, the state matrix has the following form:

$$\mathbf{A} = \begin{bmatrix} 0 & -\mathrm{M}^{-1}\mathrm{K} \\ \mathrm{I} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0{\cdot}00000000000000 & 0{\cdot}00000000000000 & -0{\cdot}28571428571429 & -0{\cdot}19632103395740 \\ 0{\cdot}00000000000000 & 0{\cdot}00000000000000 & 0{\cdot}14285714285714 & -0{\cdot}62065221321282 \\ 1.00000000000000 & 0{\cdot}00000000000000 & 0{\cdot}00000000000000 & 0{\cdot}00000000000000 \\ 0{\cdot}00000000000000 & 1.00000000000000 & 0{\cdot}00000000000000 & 0{\cdot}00000000000000 \end{bmatrix}$$

The control matrix **B** in equation (13) for single-input control force is

$$\mathbf{B} = \begin{bmatrix} 0 \\ \cdots \\ \mathbf{M}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0{\cdot}00000000000000 \\ 0{\cdot}00000000000000 \\ -0{\cdot}57142857142857 \\ 2{\cdot}28571428571429 \end{bmatrix}.$$

The eigenvalues of **A** are

$$\lambda_1 = 0{\cdot}67318886946616\mathrm{i}, \qquad \lambda_2 = 0{\cdot}67318886946616\mathrm{i},$$

$$\lambda_3 = -0{\cdot}67318886946616\mathrm{i}, \qquad \lambda_4 = -0{\cdot}67318886946616\mathrm{i},$$

where $\mathrm{i} = \sqrt{-1}$. This system is defective and eigenvalues are also the corresponding flutter frequency.

The right and left modal matrices **U** and **V** are

$$\mathbf{U} = \begin{bmatrix} -0{\cdot}46540318867956 & -0{\cdot}27308418690627 & -0{\cdot}46540318867956 & -0{\cdot}27308418690627 \\ -0{\cdot}39700584910218 & +0{\cdot}55929647822929 & -0{\cdot}39700584910218 & +0{\cdot}55929647822929 \\ 0{\cdot}69134117215010i & +0{\cdot}66056754098396\mathrm{i} & -0{\cdot}69134117215010\mathrm{i} & -0{\cdot}66056754098396\mathrm{i} \\ 0{\cdot}58973918736331\mathrm{i} & -0{\cdot}61336913663528\mathrm{i} & -0{\cdot}58973918736331\mathrm{i} & +0{\cdot}61336913663528\mathrm{i} \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} -0{\cdot}68374618664713 & -0{\cdot}53836500812307 & -0{\cdot}68374618664713 & -0{\cdot}53836500812307 \\ -0{\cdot}45788328680308 & +0{\cdot}63111613367462 & -0{\cdot}45788328680308 & +0{\cdot}63111613367462 \\ 0{\cdot}32665905473259\mathrm{i} & +0{\cdot}36242133822492\mathrm{i} & -0{\cdot}32665905473259\mathrm{i} & -0{\cdot}36242133822492\mathrm{i} \\ 0{\cdot}46489559029220\mathrm{i} & -0{\cdot}42486035051943\mathrm{i} & -0{\cdot}46489559029220\mathrm{i} & +0{\cdot}42486035051943\mathrm{i} \end{bmatrix}$$

and

$$\mathbf{P}_{m1} = \mathbf{V}_{m1}^{\mathrm{H}}\mathbf{B} = \begin{bmatrix} -0{\cdot}87595617510641\mathrm{i} \\ +1{\cdot}17820728017294\mathrm{i} \end{bmatrix},$$

$$\mathbf{P}_{m2} = \mathbf{V}_{m2}^{\mathrm{H}}\mathbf{B} = \begin{bmatrix} +0{\cdot}87595617510641\mathrm{i} \\ -1{\cdot}17820728017294\mathrm{i} \end{bmatrix}.$$

Taking the SVD of $\mathbf{P}_{m1}$ and $\mathbf{P}_{m1}$ in equation (29) yields

$$\mathbf{\Sigma}_1 = \mathrm{diag}(\sigma_1^1 = 1{\cdot}46815244976792, \qquad \sigma_2^1 = 0),$$

$$\mathbf{\Sigma}_2 = \mathrm{diag}(\sigma_1^2 = 1{\cdot}46815244976792, \qquad \sigma_2^2 = 0).$$

Since $\sigma_1^1 > 0, \sigma_2^1 = 0, \sigma_1^2 > 0, \sigma_2^2 = 0$, the first and third modes are controllable, the second and fourth modes are uncontrollable.

6. CONCLUSIONS

The vibration control of the defective system with repeated eigenvalues and a near-defective system with close eigenvalues is an important problem in actual engineering. Most of the previous discussions have focused on the non-defective system.

These results cannot be used to deal with the defective and near-defective system. The main contribution of the present paper is to extend the idea presented in reference [5] to the defective and near-defective system. The singular values of modal controllable and observable matrices can be used as the quantitative measures of modal controllability and observability of the defective modes, and the necessary and sufficient condition of controllability and observability of all the defective eigenvalues has been obtained. The analysis for modal controllability and observability of a near defective system with close eigenvalues can be transformed into one of the defective system with repeated eigenvalues, which are equal to the average value of the close eigenvalues. The results given by numerical examples show that the present procedure is valid and effective.

ACKNOWLEDGMENT

This work is supported by the National Science Foundation(19872028) and the Mechanical Technology Development Foundation of China.

REFERENCES

1. A. M. A. HAMDAN and A. H. NAYFEH 1989 *Journal of Guidance, Control, and Dynamics* **12**, 421–428. Measures of modal controllability and observability for first- and second-order linear systems.
2. B. PORTER and R. CROSSLEY 1972 *Modal Control Theory and Applications*. London: Taylor & Francis.
3. A. J. LAUB and W. F. ARNOLD 1984 *IEEE Transactions on Control AC*-**29**, 163–165. Controllability and observability criteria for multivariable linear second-order models.
4. P. C. HUGHES and R. E. SKELTON 1980 *American Society of Mechanical Engineers Journal of Applied Mechanics* **47**, 415–420. Controllability and observability of linear matrix second-order systems.
5. Z. S. LIU, D. J. WANG, H. C. HU and M. YU 1994 *Journal of Guidance, Control and Dynamics* **17**, 1377–1380. Measure of modal controllability and observability in vibration control of flexible structures.
6. A. M. A. HAMDAN 1998 *Electric-Machines-and-Power-Systems* **26**, 671–684. The SVD of system matrices and modal properties in a two-area system.
7. D. DIMOGIANOPOULOS, R. LOZANO and A. AILON 1999 *IEEE-Transactions on Automatic Control* **44**, 2308–2312. Indirect adaptive periodic control.
8. S. HERNANDEZ and A. JIMENEZ 1999 *Industrial and Engineering Chemistry Research* **38**, 3957–3963. Controllability analysis of thermally coupled distillation systems.
9. A. M. A. NUSEIRAT, A. M. A. HAMDAN and H. M. A. HAMDAN 1999 *Zeitschrift Fur Angewandte-Mathematix Und Mechanik* **79**, 473–479. Stability and modal control of an object grasped by a multifingered robot hand.
10. M. E. FURBER and D. JORDAN 1997 *Optical Engineering* **36**, 1843–1835. Optimal design of wavefront sensors for adaptive optical systems.
11. H. M. A. HAMDAM 1994 *European Transactions on Electrical Power Engineering ETEP* **4**, 61–67. Use of decentralized measures of mobility of modes for feedback signal selection and optimum siting of power system stabilizers.
12. S. H. CHEN 1999 *Matrix Perturbation Theory in Structural Dynamic Design*. Beijing: Science Press (in Chinese).
13. T. XU and S. H. CHEN 1994 *Computers and Structures* **52**, 178–185. Perturbation sensitivity of generalized modes of defective systems.
14. G. Q. SHI and D. C. ZHU 1989 *ACTA Mechanica Sinica* **21**, 212–217. Generalized modal theory for linear vibration defective systems.